JOURNAL OF APPROXIMATION THEORY 47, 1–16 (1986)

# Some Results in the Theory of Interpolation Using the Legendre Polynomial and Its Derivative

K. B. SRIVASTAVA*

*Department of Mathematics, I.I.T., Kanpur-208016 (U.P.), India*

*Communicated by Oved Shisha*

Received October 18, 1983; revised November 26, 1984

## PRELIMINARIES, NOTATIONS AND RESULTS

Let us consider a triangular matrix whose $n$th row consists of the $n$ zeros, $\{\xi_{kn}\}_{k=1}^{n}=X_0$, of the Legendre polynomial $P_n(x)$ with normalization $P_n(1)=1$ and another matrix whose $(n-1)$st row has exactly $(n-1)$ elements $\{x_{kn}\}_{k=1}^{n-1}=Y_0$ which are the zeros of $P_n'(x)$. Writing $x_k$ for $x_{kn}$, $l_k(x)$ for $l_{kn}(x)$, etc., and sometimes omitting superfluous notations, we define a so-called quasi-Hermite–Fejér interpolation process $Q_n(f; X; x)$ constructed on the set of nodes $X=X_0\cup\{-1,1\}$ (a polynomial of degree $\leqslant 2n+1$) by the following conditions:

$$Q_n(f; X; \pm 1)=f(\pm 1);\ Q_n(f; X; \xi_k)=f(\xi_k),\ Q_n'(f; X; \xi_k)=0; \quad k=\overline{1, n}. \tag{1.1}$$

Then, as is well known, $Q_n(f; X; x)$ is represented by

$$Q_n(f; X; x)=f(1)\frac{1+x}{2}P_n^2(x)+f(-1)\frac{1-x}{2}P_n^2(x) + \sum_{k=1}^{n} f(\xi_k)\frac{1-x^2}{1-\xi_k^2}l_k^2(x), \tag{1.2}$$

where $l_k(x)$, $k=\overline{1, n}$, is the fundamental Lagrange interpolation polynomial built on the set of nodes $X_0$.

The polynomial $Q_n(f; X; x)$ is known to be uniformly convergent (U.C.) if $f\in C(I)$, $I=[-1, 1]$. In regard to the rate of convergence, Prasad and

* Current address: Department of Mathematics, University of Dar es Salaam, P.O. Box 35062, Dar-es-Salaam, Tanzania.



0021-9045/86 $3.00

Varma [4] and N. Mistra [3] supplied independent pointwise estimates repectively given by

$$|Q_n(f; X; x) - f(x)| \leqslant \frac{C_1}{n} \sum_{i=1}^{n} \omega_f\left(\frac{\sqrt{1-x^2}}{i}\right) \tag{1.3}$$

and

$$|Q_n(f; X; x) - f(x)| \leqslant C_2 \sum_{i=1}^{n} \omega_f\left(\frac{(1-x^2)^{3/4}\,|P_n(x)|}{in^{1/2}}\right). \tag{1.4}$$

$C_1, C_2,\ldots$ will denote absolute positive constants independent of $n$ and $x$. $\omega_f(\cdot)$ denotes the usual modules of continuity of $f$. Generally speaking (1.3) and (1.4) are almost the same. In fact, (1.3) is a direct consequence of (1.4), and (1.4) also reflects the fact that our polynomials are interpolatory.

As we hall see later in this section, still another type of pointwise estimate, given by

$$|Q_n(f; X; x) - f(x)| = O\left[\sqrt{1-x^2}\, P_n^2(x) \sum_{i=1}^{n} \omega_f\left(\frac{\sqrt{1-x^2}}{i} + \frac{1}{i^2}\right) + \omega_f\left(\frac{\sqrt{1-x^2}\,|P_n(x)|}{n^{1/2}}\right)\right], \tag{1.5}$$

holds true. It may be mentioned that both (1.3) and (1.4) are easy conclusions of (1.5). Therefore we outline the proof of the equality (1.5) and mention other U.C. processes that may produce identical estimates. For this purpose, we start with the process $H_n(f; Y; x)$ built on the set of nodes $Y = Y_0 \cup \{-1, 1\}$ uniquely defined by the conditions

$$\begin{aligned} &H_n(f; Y; x_k) = f(x_k); \qquad H_n'(f; Y; x_k) = 0; \qquad k = \overline{1, n-1}, \\ &H_n(f; Y; \pm 1) = f(\pm 1); \qquad H_n'(f; Y; \pm 1) = 0. \end{aligned} \tag{1.6}$$

The interpolatory operator $H_n(f; Y; x)$ represented therefore by

$$\begin{aligned} H_n(f; Y; x) = {} & f(1)\left\{1 + (1-x)\frac{n(n+1)}{2}\right\}\left\{\frac{(1+x)\,P_n'(x)}{n+n+1}\right\}^2 \\ & + f(-1)\left\{1 + (1+x)\frac{n(n+1)}{2}\right\}\left\{\frac{(1-x)\,P_n'(x)}{(n+1)n}\right\}^2 \\ & + \sum_{k=1}^{n-1} f(x_k)\frac{(1-x^2)^2 P_n'^2(x)}{n^2(n+1)^2 P_n^2(x_k)(x-x_k)^2} \end{aligned} \tag{1.7}$$

will be shown to satisfy an equality of the form (1.5), viz.,

$$|H_n(f;Y;x)-f(x)| \\ = O\left[\frac{(1-x^2)^{3/2}P_n'^2(x)}{n^2}\sum_{i=1}^{n}\omega_f\left(\frac{\sqrt{1-x^2}}{i}+\frac{1}{i^2}\right)\right. \\ \left. +\,\omega_f\left(\frac{|\pi_{n+1}(x)|}{n^{3/2}}\right)\right], \tag{1.8}$$

where $\pi_{n+1}(x)=(1-x^2)\,P_n'(x)$.

Let us now take up another so-called Hermite–Féjér type of quasi-step parabola $A_{n,2}(f;X;x)$ defined uniquely by the conditions[1]

$$A_{n,2}(f;X;\xi_k)=f(\xi_k);\qquad A_{n,2}^{(\nu)}(f;X;\xi_k)=0,\qquad k=\overline{1,n};\ \nu=1,2,3, \\ A_{n,2}(f;X;\pm 1)=f(\pm 1). \tag{1.9}$$

The interpolation polynomial $A_{n,2}(f;X;x)$ is given by

$$A_{n,2}(f;X;x) \\ =f(1)\frac{1+x}{2}P_n^4(x)+f(-1)\frac{1-x}{2}P_n^4(x)+\sum_{k=1}^{n}f(\xi_k) \\ \times\left\{\frac{1-2x\xi_k+\xi_k^2}{1-\xi_k^2}+\frac{(x-\xi_k)^2}{(1-\xi_k^2)^2}\left(-\frac{1}{3}+\frac{2}{3}n(n+1)\right)(1-x\xi_k)\right\} \\ \times\frac{1-x^2}{1-\xi_k^2}l_k^4(x). \tag{1.10}$$

It is shown by A. Sharma and his associates [5] that the process $A_{n,2}(f;X;x)$ converges uniformly to $f(x)$ on $I$. We again emphasize that the equality of the type (1.5), viz.,

$$|A_{n,2}(f,x)-f(x)| \\ =O\left[n(1-x^2)\,P_n^4(x)\left\{\sum_{i=1}^{n}\omega_f\left(\frac{\sqrt{1-x^2}}{i^2}\right)+\omega_f\left(\frac{1}{i^2}\right)\right\}\right. \\ \left. +\,\omega_f\left(\frac{(1-x^2)^{1/4}\,|P_n(x)|}{n^{1/2}}\right)\right], \tag{1.11}$$

is satisfied by $A_{n,2}(f;X;x)$ also.

[1] The suffix 2 and then 4 is put in the notation for distinguishing these $A$'s from their extensions.

Exactly in the same way the $\Lambda_{n,4}(f; Y; x)$ uniquely defined by

$$\Lambda_{n,4}(f; Y; x_k) = f(x_k), \qquad \Lambda_{n,4}^{(\nu)}(f; Y; x_k) = 0; \qquad k = \overline{1, n-1},\ \nu = 1, 2, 3,$$
$$\Lambda_{n,4}(f; Y; \pm 1) = f(\pm 1), \qquad \Lambda_{n,4}^{(\nu)}(f; Y; \pm 1) = 0; \qquad \nu = 1, 2, \tag{1.12}$$

and represented by

$$\begin{aligned}\Lambda_{n,4}(f; Y; x) = {} & \frac{f(1)(1+x)^3 P_n'^4(x)}{\{n(n+1)\}^4} \Bigg\{ 2 + (n-1)(n+2)(1-x) \\ & + \frac{(n-4)(n+1)\, n(n+5)}{12} (1-x)^2 \Bigg\} \\ & + f(-1) \frac{(1-x)^3 P_n'^4(x)}{\{n(n+1)\}^4} \Bigg\{ 2 + (n-1)(n+2)(1+x) \\ & + \frac{(n-4)\, n(n+1)(n+5)}{12} (1+x)^2 \Bigg\} + \sum_{k=1}^{n-1} f(x_k) \\ & \times \left\{ \varphi_k^4(x) + \frac{2}{3} \frac{\varphi_k^4(x)(1-x_k)}{(1-x^2)\, 1 - x_k^2}\, n(n+1)(x-x_k)^2 \right\},\end{aligned} \tag{1.13}$$

$$\varphi_k(x) = \frac{\pi_{n+1}(x)}{\pi'_{n+1}(x_k)(x-x_k)} = \frac{(1-x^2)\, P_n'(x)}{n(n+1)\, P_n(x_k)(x_k - x)}, \tag{1.14}$$

may be shown to satisfy

$$\begin{aligned} & |\Lambda_{n,4}(f; Y; x) - f(x)| \\ & = O\left[ \frac{((1-x^2)^{3/4} P_n'(x))^4}{n^3} \sum_{i=1}^{n} \omega_f\left( \frac{\sqrt{1-x^2}}{i} + \frac{1}{i^2} \right) \right. \\ & \left. + \omega_f\left( \frac{(1-x^2)^{3/4} |P_n'(x)|}{n^{3/2}} \right) \right]. \end{aligned} \tag{1.15}$$

In order to illustrate the techniques of the proofs of (1.5), (1.8) (1.11), and (1.15), we now consider mixed types of the two processes $H_n$ and $\Lambda_n$ which are constructed by the following two sets of conditions on the matrix of nodes the $(2n+1)$th row of which consists of the elements of the set $Z = Y \cup X_0$:

$$H_n(f; Z; \xi_k) = f(\xi_k); \qquad H_n'(f; Z; \xi_k) = 0; \qquad k = \overline{1, n},$$
$$H_n(f; Z; x_k) = f(x_k); \qquad H_n'(f; Z; x_k) = 0; \qquad k = \overline{1, n-1}, \tag{1.16}$$
$$H_n(f; Z; \pm 1) = f(\pm 1); \qquad H_n'(f; Z; \pm 1) = 0$$

and

$$\Lambda_n(f; Z; \xi_k) = f(\xi_k); \qquad \Lambda_n(f; Z; x_k) = f(x_k),$$
$$\Lambda_n(f; Z; \pm 1) = f(\pm 1), \tag{1.17}$$
$$\Lambda_n^{(\nu)}(f; Z; \xi_k) = \Lambda_n^{(\nu)}(f; Z; x_k) = \Lambda_n^{(\nu)}(f; Z; \pm 1) = 0; \quad \nu = 1, 2, 3.$$

These processes are explicitly represented by the equalities

$$H_n(f; Z; x) = \sum_{k=1}^{n} f(\xi_k)\, h_k^{(1)}(x) + \sum_{k=1}^{n-1} f(x_k)\, h_k^{(2)}(x) + f(1)\, h_0(x) + f(-1)\, h_n(x), \tag{1.18}$$

where

$$h_n(-x) = h_0(x) = \left\{1 + \frac{3}{2} n(n+1)(1-x)\right\} \frac{(1+x)^2 P_n^2(x)\, P_n'^2(x)}{n(n+1)^2}, \tag{1.19}$$

$$h_k^{(1)}(x) = \left\{1 - \frac{2\xi_k}{1-\xi_k^2}(x-\xi_k)\right\} \frac{R_n^2(x)}{(1-\xi_k^2)^2 P_n'^4(\xi_k)(x-\xi_k)^2}; \qquad k = \overline{1, n}, \tag{1.20}$$

$$h_k^{(2)}(x) = \frac{R_n^2(x)}{n^2(n+1)^2 P_n^4(x_k)(x-x_k)^2}; \qquad k = \overline{1, n-1}, \tag{1.21}$$

$$\Lambda_n(f; Z; x) = \sum_{k=1}^{n} f(\xi_k)\, \lambda_k^{(1)}(x) + \sum_{k=1}^{n-1} f(x_k)\, \lambda_k^{(2)}(x) + f(1)\, \lambda_0(x) + f(-1)\, \lambda_n(x), \tag{1.22}$$

$$\begin{aligned}\lambda_n(-x) = \lambda_0(x) = {} & 1 + 3n(n+1)(1-x) \\ & + \frac{n(n+1)}{6}(119n^2 + 119n + 8)(1-x)^2 \\ & + \frac{5n(n+1)}{8}(349n^4 + 698n^3 + 401n^2 + 52n + 12)(1-x)^3 \\ & \times \frac{(1+x)^4 P_n^4(x)\, P_n'^4(x)}{n^4(n+1)^4},\end{aligned} \tag{1.23}$$

$$\begin{aligned}\lambda_k^{(2)}(x) = {} & \left\{1 + \frac{8n(n+1)}{3(1-x_k^2)}(x-x_k)^2 + \frac{10n(n+1)x_k}{3(1-x_k^2)^2}(x-x_k)^3\right\} \\ & \times \frac{R_n^4(x)}{n^4(n+1)^4 P_n^8(x_k)(x-x_k)^4}; \qquad k = \overline{1, n-1},\end{aligned} \tag{1.24}$$

$$\lambda_k^{(1)}(x) = \left[1 - \frac{4\xi_k(x-\xi_k)}{(1-\xi_k^2)} + (x-\xi_k)^2 \right.$$
$$\times \left\{\frac{14\xi_k^2}{3(1-\xi_\xi^2)^2} + \frac{4(2n^2+2n-1)}{3(1-\xi_k^2)}\right\}$$
$$\left. + \frac{(\xi_k - x)^3}{3}\left\{\frac{4\xi_k^3}{(1-\xi_k^2)^2} + \frac{(26n^2+26n-8)\xi_k}{(1-\xi_k^2)^2}\right\}\right]$$
$$\times \frac{R_n^4(x)}{(1-\xi_k^2)^4 P_n'^8(\xi_k)(x-\xi_k)^4}, \tag{1.25}$$
$$R_n(x) = P_n(x)\,\pi_{n+1}(x).$$

The expressions (1.22) and (1.18) are readily obtained by making use of the following relations:

$$\frac{R_n''(\xi_k)}{R_n'(\xi_k)} = \frac{2\xi_k}{1-\xi_k^2};\ \frac{R_n''(x_k)}{R_n'(x_k)} = 0;$$
$$\frac{R_n'''(\xi_k)}{R_n'(\xi_k)} = \frac{8\xi_k^2}{(1-\xi_k^2)^2} - \frac{(4n^2+4n-2)}{1-\xi_k^2};\ \frac{R_n'''(x_k)}{R_n'(x_k)} = \frac{-4n(n+1)}{1-\xi_k^2}; \tag{1.26}$$
$$\frac{R_n^{(iv)}(\xi_k)}{R_n'(\xi_k)} = \frac{48\xi_k^3}{(1-\xi_k^2)^3} - \frac{\xi_k(28n^2+28-24)}{(1-\xi_k^2)^2};\ \frac{R^{(iv)}(x_k)}{R_n'(x_k)} = \frac{-20x_k(n+1)n}{(1-x_k^2)^2};$$
$$\frac{R_n''(1)}{R_n'(1)} = \frac{3}{2}n(n+1) = -\frac{R_n''(-1)}{R_n'(-1)};$$
$$\frac{R_n'''(1)}{R_n'(1)} = \frac{n(n+1)}{2}(2n^2+2n-1) = \frac{R_n'''(-1)}{R_n'(-1)}; \tag{1.27}$$
$$\frac{R_n^{(iv)}(1)}{R_n'(1)} = \frac{5(n-1)\,n(n+1)(n+2)}{48}(7n^2+7n-6) = -\frac{R_n^{(iv)}(-1)}{R_n'(-1)}.$$

The relation (1.26) and (1.27) can be verified by the differential equations satisfied, respectively, by $P_n(x)$ and $\pi_{n+1}(x)$,

$$\begin{aligned}(1-x^2)\,P_n''(x) - 2xP_n'(x) + n(n+1)\,P_n(x) &= 0,\\ (1-x^2)\,\pi_{n+1}''(x) + n(n+1)\,\pi_{n+1}(x) &= 0,\end{aligned} \tag{1.28}$$

and by the following relations:

$$P_n'(1) = \tfrac{1}{2}n(n+1) = (-1)^{n-1}P_n(-1);$$
$$P_n''(1) = \tfrac{1}{8}(n+2)(n+1)\,n(n-1) = (-1)^n P_n''(-1);$$

$$\begin{aligned}
P_n'''(1) &= \tfrac{1}{48}(n+3)(n+2)(n+1)(n)(n-1)(n-2)\\
&= (-1)^{n-1}P_n'''(-1);\\
P_n^{(\mathrm{iv})}(1) &= \frac{1}{384}(n+4)(n+3)(n+2)(n+1)\,n(n-1)(n-2)(n-3)\\
&= (-1)^n P_n^{(\mathrm{iv})}(-1);\\
\pi_{n+1}'(1) &= -n(n+1) = (-1)^{n-1}\pi_{n+1}'(-1);\\
\pi_{n+1}''(1) &= -\tfrac{1}{2}n^2(n+1)^2 = (-1)^n\pi_{n+1}''(-1);\\
\pi_{n+1}'''(1) &= -\tfrac{1}{8}(n+2)(n-1)\,n^2(n+1)^2 = (-1)^{n-1}\pi_n'''(-1);\\
\pi_{n+1}^{(\mathrm{iv})}(1) &= -\tfrac{1}{48}n^2(n+1)^2(n+3)(n+2)(n-1)(n-2)\\
&= (-1)^n\pi_{n+1}^{(\mathrm{iv})}(-1).
\end{aligned} \tag{1.29}$$

Finally, the process $A_n(f; x)$ defined for any set of nodes $\{x_k\}_{k=1}^{n}$ that are the zeros of orthogonal polynomial $W_n(x)$ is represented by

$$A_n(f; x) = \sum_{k=1}^{n} f(x_k)\, u_k(x)\, L_k^4(x), \tag{1.30}$$

where

$$\begin{aligned}
u_k(x) = 1 - 2(x-x_k)\frac{W_n''(x_k)}{W_n'(x_k)}
&+ \frac{(x-x_k)^2}{2}\left\{5\left[\frac{W_n''(x_k)}{W_n'(x_k)}\right]^2 - \frac{4}{3}\frac{W_n'''(x_k)}{W_n'(x_k)}\right\}\\
&+ \frac{(x-x_k)^3}{6}\left\{-15\left[\frac{W_n''(x_k)}{W_n'(x_k)}\right]^3\right.\\
&\left.+10\frac{W_n''(x_k)\,W_n'''(x_k)}{[W_n'(x_k)]^2} - \frac{W_n^{(\mathrm{iv})}(x_k)}{W_n'(x_k)}\right\};
\end{aligned} \tag{1.31}$$

$$L_k(x) = \frac{W_n(x)}{W_n'(x_k)(x-x_k)}, \qquad k = \overline{1, n}. \tag{1.32}$$

We now state the main result of this section as follows:

THEOREM 1. *Let $H_n(f; Z; x)$ and $A_n(f; Z; x)$ be represented by* (1.18) *and* (1.22) *and $f \in C(I)$; then for each $x \in I$ and every natural number $n$, we have*

(a) $|H_n(f; Z; x) - f(x)|$

$$= O\left[\frac{R_n^2(x)}{n} \sum_{i=1}^{n} \omega_f\left(\frac{\sqrt{1-x^2}}{i} + \frac{1}{i^2}\right) + \omega_f\left(\frac{|R_n(x)|}{n}\right)\right]$$

*and*

(b) $|A_n(f; Z; x) - f(x)|$

$$= O\left[\frac{R_n^4(x)}{n} \sum_{i=1}^{n} \omega_f\left(\frac{\sqrt{1-x^2}}{i} + \frac{1}{i^2}\right) + \omega_f\left(\frac{|R_n(x)|}{n}\right)\right].$$

In order the establish the theorem we need a series of results given in the forms of the following lemmas:

LEMMA 1. *For each $x \in I$ and $n \geqslant 3$, we have*

$$h_k^{(1)}(x) \leqslant \frac{C_{11}}{i_1^2}, \qquad k \neq j_1; |k - j_1| = i_1, \tag{1.33}$$

$$h_k^{(2)}(x) \leqslant \frac{C_{12}}{i_2^2}, \qquad k \neq j_2; |k - j_2| = i_2, \tag{1.34}$$

$$\lambda_k^{(1)}(x) \leqslant \frac{C_{13}}{i_1^2}, \tag{1.35}$$

$$\lambda_k^{(2)}(x) \leqslant \frac{C_{14}}{i_2^2}, \tag{1.36}$$

*where $j_1$ and $j_2$ are defined by*

$$|x - \xi_{j_1}| = \min_{1 \leqslant k \leqslant n} |x - \xi_k| \tag{1.37}$$

*and*

$$|x - x_{j_2}| = \min_{1 \leqslant k \leqslant n-1} |x - x_k|. \tag{1.38}$$

LEMMA 2. *We have, for each $x \in I$ and $j_1, j_2$ defined by* (1.37) *and* (1.38),

$$|x - \xi_{j_1}|\, h_{j_1}^{(1)}(x) \leqslant C_{15} \frac{|R_n(x)|}{n}; \tag{1.39}$$

$$|x - x_{j_2}|\, h_{j_2}^{(2)}(x) \leqslant C_{16} \frac{|R_n(x)|}{n}; \tag{1.40}$$

$$|x - \xi_{j_1}|\, \lambda_{j_1}^{(1)}(x) \leqslant C_{17} \frac{|R_n(x)|}{n} \tag{1.41}$$

*and*

$$|x-x_{j_2}|\,\lambda_{j_2}^{(2)}(x)\leqslant C_{18}\frac{|R_n(x)|}{n}. \tag{1.42}$$

LEMMA 3 [O. Kis and P. Vertesi]. *Let* $x=\cos\theta$, $\xi_k=\cos\theta_k$, $x_k=\cos\Psi_k$. *Then*

$$\omega_f(|x-\xi_k|)=O\left[\omega_f\left(\frac{i_1\sqrt{1-x^2}}{n}+\frac{i_1^2}{n^2}\right)\right]; \qquad k\neq j_1,$$

$$=O\left[\omega_f\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)\right]; \qquad k=j_1.$$

*Similar estimates hold for* $\omega_f[|x-x_k|)$.

LEMMA 4. *The following estimates are valid*:

(i) $(1-x^2)^{1/4}\,|P_n(x)|\leqslant\frac{\sqrt{2}}{\pi}n^{-1/2}$,

(ii) $(1-x^2)^{3/4}\,|P_n'(x)|\leqslant\frac{\sqrt{8}}{\pi}n^{1/2}$,

(iii) $(1-x^2)^{1/2}\,|P_n'(x)|\leqslant n$,

(iv) $|P_n(x)|\leqslant 1$,

(v) $|R_n(x)|\leqslant 2$,

*for* $x\in I$, $n\geqslant 3$,

(vi) $1-\xi_k^2>\left(k-\frac{1}{2}\right)^2\left(n+\frac{1}{2}\right)^{-2}$; $\qquad k=\overline{1,\left[\frac{n}{2}\right]}$,

$>\left(n-k+\frac{1}{2}\right)^2\left(n+\frac{1}{2}\right)^{-2}$; $\qquad k=\overline{\left[\frac{n}{2}\right]+1,n}$,

(vii) $P_n'(\xi_k)\sim k^{-3/2}n^2$; $k=\overline{1,\left[\frac{n}{2}\right]}$,

$\sim(n-k+1)^{-3/2}n^2$, $\qquad k=\overline{\left[\frac{n}{2}\right]+1,n}$,

(viii) $\frac{(k-\frac{1}{2})\pi}{n+\frac{1}{2}}<\theta_k<\frac{k\pi}{n+\frac{1}{2}}$; $\qquad k=\overline{1,n}$,

(ix) $|P_n(x_k)|\geqslant\sqrt{\frac{1}{8k\pi}}$, $\qquad k=\overline{1,n-1}$,

$$\text{(x)}\quad n\sqrt{1-x_k^2}\,P_n^2(x_k) \geqslant \frac{1}{8\pi},$$

$$\text{(xi)}\quad (1-\xi_k^2)^{3/2} P_n'^2(\xi_k) \geqslant C_1 n; \qquad k=\overline{1,n},$$

$$\text{(xii)}\quad 1 = x_0 > \xi_1 > x_1 > \xi_2 > x_2 > \xi_3 > \cdots > \xi_{n-1} > x_{n-1} > \xi_n > x_n = -1.$$

LEMMA 5. *The following identities hold:*

$$\text{(i)}\quad \sum_{k=1}^{n} \frac{1}{(1-\xi_k^2)\,P_n'^2(\xi_k)} = 1,$$

$$\text{(ii)}\quad \sum_{k=1}^{n} \frac{1}{1-\xi_k} = \sum_{k=1}^{n} \frac{1}{1+\xi_k} = \sum_{k=1}^{n} \frac{1}{1-\xi_k^2} = P_n'(1),$$

$$\text{(iii)}\quad \sum_{k=1}^{n} \frac{1}{(1-\xi_k)^2} = P_n'^2(1) - P_n''(1),$$

$$\text{(iv)}\quad \sum_{k=1}^{n} \frac{1-x^2}{1-\xi_k^2}\,\xi_k^2(x) = 1 - P_n^2(x),$$

$$\text{(v)}\quad \sum_{k=1}^{n} \frac{1}{(1-\xi_k)^3(1+\xi_k)\,P_n'^2(\xi_k)} = P_n'(1),$$

$$\text{(vi)}\quad \sum_{k=1}^{n-1} \frac{1}{(1-x_k^2)\,P_n^4(x_k)}\,\frac{1}{1-x_k} = O(n^4).$$

LEMMA 6. *The following relation for the modulus of continuity of $f$ is valid for $x \in I$:*

$$\sum_{i=1}^{n} \frac{1}{i^2}\left\{\omega_f\left(\frac{i\sqrt{1-x^2}}{n}\right) + \omega_f\left(\frac{i^2}{n^2}\right)\right\}$$
$$= O\left(\frac{1}{n}\right) \sum_{i=1}^{n} \left[\omega_f\left(\frac{\sqrt{1-x^2}}{i}\right) + \omega_f\left(\frac{1}{i^2}\right)\right].$$

Proof of the various parts of the Lemma 5 can either be found in [4, 5] or can easily be established by making use of the definitions and results already known. For example, parts (ii) and (iii) are easy consequences of

$$P_n'(x) = \sum_{k=1}^{n} \frac{P_n(x)}{x-\xi_k}, \tag{1.43}$$

$$P_n'^2(x) - P_n''(x)\,P_n(x) = \sum_{k=1}^{n} \frac{P_n^2(x)}{(x-\xi_k)^2}. \tag{1.44}$$

Lemma 6 has already been shown in the book of R. A. DeVore [1]. Most of the results stated in Lemma 4 can be found in any standard text on orthogonal polynomials (e.g., Szegö [6]). The rest were given by P. Turán, P. Erdös, and others. Lemma 3 is self-evident.

In order to prove Lemma 2 (say, (1.39)), we set

$$l_k^{(1)}(x) = \frac{R_n(x)}{(1-\xi_k^2)\, P_n'^2(\xi_k)(x-\xi_k)}, \tag{1.45}$$

$$l_k^{(2)}(x) = \frac{R_n(x)}{n(n+1)\, P_n^2(x_k)(x_k - x)}. \tag{1.46}$$

We obtain the following identities from the representations (1.22) and (1.18), putting $f(t) \equiv 1$:

$$1 \equiv h_0(x) + h_n(x) + \sum_{k=1}^{n} h_k^{(1)}(x) + \sum_{k=1}^{n-1} h_k^{(2)}(x), \tag{1.47}$$

$$1 \equiv \lambda_0(x) + \lambda_n(x) + \sum_{k=1}^{n} \lambda_k^{(1)}(x) + \sum_{k=1}^{n-1} \lambda_k^{(2)}(x). \tag{1.48}$$

It is clear from (1.19–(1.21) that

$$h_k^{(1)}(x) > 0, \qquad h_k^{(2)}(x) > 0, \qquad h_n(-x) = h_0(x) > 0,$$

which imply that

$$\sum_{k=1}^{n} h_k^{(1)}(x) \leqslant 1.$$

From the above relation we have

$$\sum_{k=1}^{n} \left[\frac{(1-x^2)+(x-\xi_k)^2}{(1-\xi_k^2)}\right] l_k^{(1)2}(x) \leqslant 1$$

or

$$\sum_{k=1}^{n} \frac{1-x^2}{1-\xi_k^2}\, l_k^{(1)2}(x) \leqslant 1 - \frac{C_2\, |R_n(x)|}{n}, \qquad C_2 < 1,$$

which, in turn, implies

$$\frac{\sqrt{1-x^2}}{1-\xi_k^2}\, |l_k^{(1)}(x)| \leqslant C_3, \qquad k = \overline{1, n}. \tag{1.49}$$

Now, owing to the representations

$$
\begin{aligned}
|x-x_{j_1}|\, h_{j_1}^{(1)}(x) &= \frac{\{(1-x^2)+(x-\xi_k)^2\}}{1-\xi_k^2}\, |x-\xi_{j_1}|\, l_{j_2}^{(1)2}(x) \\
&\leqslant \frac{C_2 R_n^2(x)}{n^2} + \frac{|R_n(x)|\,(1-x^2)\,|l_{j_1}^{(1)}(x)|}{(1-\xi_{j_1}^2)^2\, P_n'^2(\xi_{j_1})} \\
&\leqslant \frac{C_2 P_n^2(x)}{n^2} + \frac{C_4\,|R_n(x)|}{n} \\
&\frac{\leqslant C_5\,|R_n(x)|}{n},
\end{aligned}
$$

proving thereby (1.39).

Quite similarly it can be shown that

$$|l_k^{(2)}(x)| \leqslant C_6, \qquad k=\overline{1, n-1}.$$

Therefore from the inequality obtained from (1.47), i.e.,

$$\sum_{k=1}^{n-1} l_k^{(2)^2}(x) \leqslant C_7,$$

and the relations

$$
\begin{aligned}
|x-\xi_{j_2}|\, h_{j_2}^{(2)}(x) &= \frac{R_n^2(x)}{n^2(n+1)^2 P_n^4(x_{j_2})\,|x-x_{j_2}|} \\
&\leqslant \frac{|R_n(x)|}{n} \sqrt{1-x_{j_2}^2}\, \frac{R_n^2(x)}{n}\, |l_{j_2}^{(2)}(x)| \\
&\leqslant \frac{C_8\,|R_n(x)|}{n},
\end{aligned}
$$

the assertion (1.40) follows at once.

We observe from the identity (1.48) that

$$\lambda_k^{(1)}(x) > 0, \qquad \lambda_k^{(2)}(x) > 0, \qquad \lambda_0(x) = \lambda_n(-x) > 0$$

due to which

$$\sum_{k=1}^{n} \lambda_k^{(1)}(x) \leqslant 1.$$

Owing to this inequality, we immediately have

$$\sum_{k=1}^{n} \frac{(1-x^2)^2}{(1-\xi_k^2)^2} l_k^{(1)4}(x) \leqslant C_9,$$

which gives

$$\begin{aligned}
&|x-\xi_{j_1}|\, \lambda_{j_1}^{(1)}(x) \\
&= |x-\xi_1| \left\{ \frac{(1-x^2)^2}{1-\xi_{j_1}^2} + \frac{4}{3} \frac{(1-x^2)(x-\xi_{j_1})^2}{(1-\xi_{j_1}^2)^2} \right. \\
&\quad + \frac{(x-\xi_{j_1})^4}{(1-\xi_{j_1}^2)^2} + \frac{2}{3} \frac{(1-x^2)(x-\xi_{j_1})^2}{(1-\xi_{j_1}^2)^3} + \frac{2}{3} \frac{(x-\xi_{j_1})^4}{(1-\xi_{j_1}^2)^3} \\
&\quad + \frac{4(2n^2+2n-1)(x-\xi_{j_1})^2}{3(1-\xi_{j_1}^2)^3} \{(1-x^2)+(x-\xi_{j_1})^2\} \\
&\quad \left. - \frac{10n(n+1)\, \xi_{j_1}}{3} \frac{(x-\xi_{j_1})^3}{(1-\xi_{j_1}^2)^2} \right\} l_{j_1}^{(1)}(x) \\
&\leqslant C_{10} \frac{|R_n(x)|}{n}. \qquad (1.50)
\end{aligned}$$

The same arguments lead to (1.44) also. This completes the proof of Lemma 2.

We now show the various parts of Lemma 1. Owing to

$$\begin{aligned}
h_k^{(1)}(x) &= \frac{[(1-x^2)+(x-\xi_k)^2]\, R_n^2(x)}{(1-\xi_k^2)^3 P_n'^4(\xi_k)(x-\xi_k)^2} \\
&= \frac{(1-x^2)\, R_n^2(x)}{(1-\xi_k^2)^3 P_n'^4(\xi_k)(x-\xi_k)^2} + \frac{R_n^2(x)}{(1-\xi_k^2)^3 P_n'^4(\xi_k)} \\
&\leqslant \frac{R_n^2(x) \sin^2\theta}{C_1^2 n^2\, 4 \sin^2 \dfrac{\theta+\theta_k}{2} \sin^2 \dfrac{\theta-\theta_k}{2}} + \frac{R_n^2(x)}{C_1^2 n^2} \\
&\leqslant \frac{R_n^2(x)}{C_1^2 n^2 \sin^2 \dfrac{\theta-\theta_k}{2}} + \frac{R_n^2(x)}{C_1^2 n^2} \\
&\leqslant \frac{2R_n^2(x)}{C_1^2 n^2 \sin^2 \dfrac{\theta-\theta_k}{2}}
\end{aligned}$$

and the inequality

$$\sin\frac{|\theta-\theta_k|}{2}\geqslant\frac{C_{11}i_1}{n},\qquad i_1=|k-j_1|;k\neq j_1,\tag{1.51}$$

which is evident by making use of suitable forms of Lemma 4, we immediately obtain (1.33). The inequality (1.34) can be proved similarly. Owing to (1.46) we get (1.36) if we take into account (1.24) and

$$\sin\frac{|\theta-\Psi_k|}{2}\geqslant\frac{C_{12}i_2}{n},\qquad i_2=|k-j_2|;k\neq j_2.\tag{1.52}$$

To show (1.35), we make use of (1.51) and (1.25) and the representation (1.48). Thus we have completely proved Lemma 1.

To prove our main result, we take into account the identity (1.47) and write the difference

$$\begin{aligned}H_n(f;Z;x)-f(x)&=[f(1)-f(x)]\,h_0(x)+[f(-1)-f(x)]\,h_n(x)\\&\quad+\sum_{k=1}^{n}[f(\xi_k)-f(x)]\,h_k^{(1)}(x)\\&\quad+\sum_{k=1}^{n-1}[f(x_k)-f(x)]\,h_k^{(2)}(x)\\&=\sum_{p=1}^{4}T_{p'}\text{ say.}\end{aligned}\tag{1.53}$$

To conclude the proof of the theorem, we show that

$$\begin{aligned}|T_1|+|T_2|&\leqslant[(1+x)\,\omega_f(1-x)+(1-x)\,\omega_f(1+x)]\\&\quad\times\left[\frac{2P_n^2(x)\,P_n'^2(x)}{n^2(n+1)^2}+\frac{3}{2}\frac{(1-x^2)\,P_n^2(x)\,P_n'^2(x)}{n(n+1)}\right],\end{aligned}$$

which, owing to simple properties of the modulus of continuity, at once gives

$$\begin{aligned}|T_1|+|T_2|&\leqslant 24\omega_f\left(\frac{|R_n(x)|}{n}\right)+18\omega_f\left(\frac{|R_n(x)|}{n}\right)\\&=42\omega_f\left(\frac{|R_n(x)|}{n}\right).\end{aligned}$$

For $T_3$ and $T_4$, the patterns are similar. Therefore we work out only one of them. For instance, $T_3$ is first broken into two parts, viz.,

$$T_3 = T_{31} + T_{32},$$

where

$$T_{31} = \sum_{\substack{k=1 \\ k \neq j_1}}^{n} [f(x_k) - f(x)]\, h_k^{(1)}(x) \tag{1.54}$$

and

$$T_{32} = [f(x_{j_1}) - f(x)]\, h_{j_1}^{(1)}(x). \tag{1.55}$$

We shall now show that

$$|T_{32}| \leqslant C_{15}\omega_f\left(\frac{|R_n(x)|}{n}\right), \tag{1.56}$$

which is a direct conclusion of suitable forms of Lemma 2 and the inequality

$$\sum_{i=1}^{n} h_k^{(1)}(x) \leqslant 1.$$

Using the appropriate forms of Lemma 1, we have the desired form for $T_{31}$, i.e.,

$$|T_{31}| \leqslant R_n^2 C_{16} \sum_{i_1=1}^{n} \left[\omega_f\left(\frac{i_1\sqrt{1-x^2}}{n}\right) + \omega_f\left(\frac{1}{i_1^2}\right)\right]. \tag{1.57}$$

Now applying Lemma 6, we get the required form. Because of (1.53)–(1.57) we have our theorem once the estimates for $T_4$ are also computed.

Similar arguments apply to part (b).

## Acknowledgment

The author thanks the referee for valuable suggestions.

## References

1. R. A. DeVore, "The Approximation of Continuous Functions by Positive Linear Operators," Springer-Verlag, New York, 1972.

2. T. M. MILLS AND S. J. GOODENOUGH, On interpolation polynomials of the Hermite–Fejér type, II, *Bull. Austral. Math. Soc.* **23** (1981), 283–291.
3. N. MISRA, On the rapidity of convergence of Hermite–Fejér interpolation based on the roots of Legendre polynomials, *Acta Math Acad. Sci. Hungar.* **39**, Nos. 1–3 (1982), 149–154.
4. J. PRASAD AND A. K. VERMA, A study of some interpolating processes based on the roots of Legendre polynomials, *J. Approx. Theory* **31** (1981), 244–252.
5. A. SHARMA AND J. TZIMBALIRIO, Quasi-Hermite–Fejér type interpolation of higher order, *J. Approx. Theory* **13** (1975), 431–442.
6. G. SZEGÖ, "Orthogonal Polynomials," Amer. Math. Soc. Publ., Vol. 23, Amer. Math. Soc., Providence, R.I., 1978.